A numerical analysis of variational valuation techniques for derivative securities

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Abstract

In this paper, we consider the partial differential equations approach for valuing European and American style options on multiple assets. We use a method of lines finite element implementation available in the software package FEMLAB in order to solve the variational inequality that characterizes the American style option, as well as the partial differential equation that defines the European style option, for two and three state variables. A detailed study of the approximation error is provided, including a theoretical estimate, an asymptotic analysis, the space–time distribution, and the dependence on the size of the truncation domain.

Keywords: Mathematical finance; Variational inequalities; Finite element discretization; Option pricing

1. Introduction

At the most fundamental level, derivative securities are defined as stochastic control problems involving a system of stochastic differential equations and a
payoff function. In the presence of early exercise features, the value function of the security is generally represented deterministically. From this genesis, the valuation process may be analyzed equivalently from either a probabilistic or variational perspective. These two approaches may be reconciled by observing that the exact same discretization may be obtained through the formal application of the dynamic programming principal to the stochastic system or the numerical approximation of the variational problem (as in [11]).

Irrespective of the discretization paradigm, the computational efficiencies are a consequence of the approximation order, optimal mesh generation, and an efficient solver. Robust error estimation is also desirable and can relate to the dynamic evolution of the method. Indeed, deterministic numerical techniques for valuing derivative securities remain in their infancy relative to other applications within the engineering sciences. As such, this paper seeks to establish the basic formalism of the problem, to which more advanced techniques can be subsequently applied. Without loss of generality, we choose a variational representation owing to the availability of applicable production software. In this regard, we present both theoretical and computational convergence results relative to a baseline method of lines implementation.

In Section 2, we consider the representation of the value function as the solution of a variational inequality and its approximation by a nonlinear equation through a penalization technique. In Section 3, the continuous penalization problem is approximated spatially by the finite element method using the software package FEMLAB, resulting in a system of ordinary differential equations. Considerations relative to the numerical solutions of this stiff system of ordinary differential equations are presented, as well as estimates pertaining to the quality of the approximation. In particular, the role of the computational and approximation domains are detailed, as well as the nature of the artificial boundary conditions. In Section 4, we present results of numerical experiments, for both European and American style options, involving two and three assets. The role of the penalization parameter and the mesh/time scaling necessary for asymptotic convergence is detailed. We consider the distribution of spatial error on the approximation region for a fixed time as well as the evolution of spatial error with time. In particular, the initial transient region is examined as is the global corrupting influence of the artificial boundary conditions for large time scales. Finally, we show how the size of the computational domain can be computed for a given duration, accuracy and approximation domain size.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\mathcal{F}$, and $(B_t)_{t \geq 0}$ be a standard Brownian motion with values in $\mathbb{R}^n$. On $\Omega$, we consider $n$
risky assets, with prices at time \( \tau \) \( S_{\tau,i}, i = 1, \ldots, n \), governed by the following stochastic differential equation

\[
\frac{dS_{\tau,i}}{S_{\tau,i}} = r \, d\tau + \sigma_i \, dB_{\tau,i}, \quad i = 1, \ldots, n
\tag{2.1}
\]

on a finite interval \([0, T]\), where \( T \) is the time horizon (expiry). In (2.1), \( r \in \mathbb{R}_+ \) represents the drift, and \( \sigma_i \in \mathbb{R}_+ \) represents the volatility for each asset \( S_{\tau,i} \).

Associated with the flow (2.1), and with a given payoff function \( \psi \), the value of an European option is given by the expected value of the depreciated payoff [10]

\[
u(S_{\tau}, \tau) = E_p(e^{-r(T-\tau)}\psi(S_T)),
\tag{2.2}
\]

while, for an American option, we seek to find the optimal solution of the stopping problem

\[
u(S_{\tau}, \tau) = \sup_{\theta \in \mathcal{F}_{[\tau,T]}} E_p(e^{-r(T-\tau)}\psi(S_{\theta})),
\tag{2.3}
\]

where \( \mathcal{F}_{[\tau,T]} \) represents the set of stopping times in \([\tau,T]\).

Under the hypothesis of a no arbitrage market, the price of an European option can also be expressed as the solution of the following initial value problem (see [10, Chapter 5])

\[
\frac{\partial u}{\partial t} - \mathcal{A}u = 0 \quad \text{on} \quad \mathbb{R}^n \times (0,T],
\tag{2.4a}
\]

\[
u(x,0) = \psi(x) \quad \text{on} \quad \mathbb{R}^n
\tag{2.4b}
\]

in which \( \mathcal{A} \) represents the elliptic operator

\[
\mathcal{A}u = \sum_{i=1}^{n} \left[ \frac{1}{2} \sigma_i^2 \frac{\partial^2 u}{\partial x_i^2} + \left( r - \frac{1}{2} \sigma_i^2 \right) \frac{\partial u}{\partial x_i} \right] - ru.
\tag{2.5}
\]

In obtaining (2.4) and (2.5) from (2.1), we have performed the change of variables \( x_i = \log(S_i) \) and \( t = T - \tau \).

Many different types of payoff are available in the trading of options. We consider the three different types:

- the payoff corresponding to a call option on the minimum of \( n \) assets, given by
  \[
  \psi(x) = \max(\min(e^{x_1}, \ldots, e^{x_n}) - E, 0);
  \tag{2.6}
  \]

- the payoff corresponding to a put option on the minimum of \( n \) assets, given by
  \[
  \psi(x) = \max(E - \min(e^{x_1}, \ldots, e^{x_n}), 0);
  \tag{2.7}
  \]

and
- the payoff corresponding to a put option on the geometric average of $n$ assets
  \[ \psi(x) = \max\{E - (e^{x_1} \ldots e^{x_n})^{1/n}, 0\}. \]  

(2.8)

In (2.6)–(2.8), $E$ represents the prescribed amount for which the buyer of the option may exercise the option at expiry, called the exercise price. In the present framework, the price of an European option is determined by the unique solution of (2.4). For a given payoff of one of the forms above, using the Green function solution representation (see [7, Section 10.3.7]), we can solve (2.4) directly, in order to find the price of the European option on $\mathbb{R}^n \times (0, T]$.

From a computational point of view, we will discretize $\mathbb{R}^n$ into an increasing sequence of bounded domains $\{\Omega_k\}_{k=1}^{\infty}$, such that $\lim_{k \to \infty} \Omega_k = \mathbb{R}^n$. Therefore, we will computationally solve (2.4) on $\Omega_k \times (0, T]$, i.e. we solve

\[
\frac{\partial u_k}{\partial t} - \mathcal{A} u_k = 0 \quad \text{on } \Omega_k \times (0, T],
\]

(2.9a)

\[ u_k(x, 0) = \psi(x) \quad \text{on } \Omega_k. \]

(2.9b)

Since the boundary conditions as $x \to \infty$ can be approximated by $u(x, t) = \psi(x)$, then consider (2.9) subject to

\[ u_k(x, t) = \psi(x) \quad \text{on } \Gamma_k \times (0, T], \]

(2.10)

where $\Gamma_k = \partial \Omega_k$.

The price of an American option is governed by the variational inequality [10, Section 5.3]

\[
\frac{\partial u}{\partial t} - \mathcal{A} u \geq 0 \quad \text{on } \mathbb{R}^n \times (0, T],
\]

(2.11a)

\[ u \geq \psi \quad \text{on } \mathbb{R}^n \times (0, T], \]

(2.11b)

\[
\left( \frac{\partial u}{\partial t} - \mathcal{A} u \right)(u - \psi) = 0 \quad \text{on } \mathbb{R}^n \times (0, T]
\]

(2.11c)

subject to (2.4b), where $\mathcal{A}$ and $\psi$ are defined by (2.5)–(2.8). As with an European option, we compute the solution of (2.11) on a bounded domain $\Omega_k \times (0, T]$, i.e.

\[
\frac{\partial u_k}{\partial t} - \mathcal{A} u_k \geq 0 \quad \text{on } \Omega_k \times (0, T],
\]

(2.12a)

\[ u_k \geq \psi \quad \text{on } \Omega_k \times (0, T], \]

(2.12b)

\[
\left( \frac{\partial u_k}{\partial t} - \mathcal{A} u_k \right)(u_k - \psi) = 0 \quad \text{on } \Omega_k \times (0, T],
\]

(2.12c)

with (2.9b) and (2.10) satisfied.

It is shown [2, Section 3.4.9; 10, Section 5.2.1] that for any bounded domain $\bar{G} \subset \Omega_k$...
\[
\lim_{k \to \infty} \left( \max_{t \in [0,T]} \| u(x,t) - u_k(x,t) \|_{L^\infty(\Omega)} \right) = 0, \tag{2.13}
\]
where \( u(x,t) \) is the solution of (2.4b) and (2.11), and \( u_k(x,t) \) is the solution of (2.9b), (2.10), and (2.12). We will verify the above estimate numerically in Section 4, for some particular choices of \( G \).

The variational inequality (2.12) can be simplified to a nonlinear partial differential equation, by using the penalization technique ([16], also in [2, Section 3.2.3; 6, Section 1.7]). We present the main steps used in applying this method. Let \( V \) be the Hilbert space \( L^1([0,T]; W^{1,2}(\Omega_k)) \) and introduce the convex and closed subset
\[
K_k = \{ v \in V / v \geq \psi \text{ on } \Omega_k \} \tag{2.14}
\]
By multiplying (2.12a) by \( v - \psi \geq 0 \), where \( v \) is an arbitrary test function in \( K_k \) and integrating on \( \Omega_k \) we obtain
\[
\left( \frac{\partial u_k}{\partial t} - \mathcal{A} u_k, v - \psi \right)_{\Omega_k} \geq 0, \quad \forall v \in K_k, \tag{2.15}
\]
where \((\cdot, \cdot)_{\Omega_k}\) is the usual \( L^2(\Omega_k) \) inner product
\[
(f, g)_{\Omega_k} = \int_{\Omega_k} f(x)g(x) \, dx, \quad \forall f, g \in L^2(\Omega_k). \tag{2.16}
\]
Integrating (2.12c) on \( \Omega_k \), we find
\[
\left( \frac{\partial u_k}{\partial t} - \mathcal{A} u_k, u_k - \psi \right)_{\Omega_k} = 0. \tag{2.17}
\]
Subtracting (2.17) from (2.15), we have
\[
\left( \frac{\partial u_k}{\partial t} - \mathcal{A} u_k, v - u_k \right)_{\Omega_k} \geq 0, \quad \forall v \in K_k. \tag{2.18}
\]
The unique solution of (2.18) such that (2.12b) is satisfied is a solution of the original variational inequality (2.12) (see [2, Remark 2.3, p. 236]).

Thus, in order to approximately satisfy the constraint (2.12b) for (2.18), we introduce the functional \( j(\cdot) : V \rightarrow \mathbb{R} \) as
\[
j(v) = \frac{1}{2} \min_{\Omega_k} (v - \psi, 0)^2. \tag{2.19}
\]
We remark that \( j \) is a convex, lower semicontinuous and proper functional on \( V \), with the additional desirable properties
\[
j(v) = 0, \quad \forall v \in K_k, \tag{2.20a}
\]
\[
j(v) \geq 0, \quad \forall v \in V - K_k. \tag{2.20b}
\]
Therefore, \( j \) is a penalty function with respect to the constraint (2.12b).
Let us also define $j_{\varepsilon}(\cdot) : V \rightarrow \mathbb{R}$, as

$$j_{\varepsilon}(v) = \frac{1}{\varepsilon} j(v), \quad \forall v \in V,$$  \hspace{1cm} (2.21)

where $\varepsilon$ is a small parameter.

By considering both cases $u_{ke} \in K_k$ and $u_{ke} \in V - K_k$ separately, it can be shown that the solution of (2.12b) and (2.18) is approximated by the solution of

$$
\left( \frac{\partial u_{ke}}{\partial t} - \mathcal{A}_{uk_{ke}, v - u_{ke}} \right)_{\Omega_k} + j_{\varepsilon}(v) - j_{\varepsilon}(u_{ke}) \geq 0, \quad \forall v \in K_k,
$$  \hspace{1cm} (2.22)

since $j_{\varepsilon}(v) = 0$ (see (2.20a)). Intuitively, this approximation improves as $\varepsilon$ becomes smaller, since this would add a tighter penalization term to (2.18).

When $j_{\varepsilon}$ is differentiable on $V$, (2.22) can be written as

$$
\left( \frac{\partial u_{ke}}{\partial t} - \mathcal{A}_{uk_{ke}, v - u_{ke}} \right)_{\Omega_k} + j'_{\varepsilon}(u_{ke}) (v - u_{ke}) \geq 0, \quad \forall v \in K_k,
$$  \hspace{1cm} (2.23)

which is the unique solution of

$$
\frac{\partial u_{ke}}{\partial t} - \mathcal{A}_{uk_{ke}} + j'_{\varepsilon}(u_{ke}) = 0
$$  \hspace{1cm} (2.24)

(see [6, Remark 7.1, p. 16]).

For $j(\cdot)$ introduced by (2.19), $j'_{\varepsilon}$ is given by

$$
j'_{\varepsilon}(v) = \frac{1}{\varepsilon} \min(v - \psi, 0), \quad \forall v \in V.
$$  \hspace{1cm} (2.25)

The penalized form (2.24) represents an approximation of the original variational inequality (2.12).

**Remark 2.1.** The following error estimate holds

$$
\max_{t \in [0, T]} \| u_{ke}(x, t) - u_{k}(x, t) \|_{L^\infty(\Omega_k)} \leq C(\varepsilon) \quad \text{as} \quad \varepsilon \to 0,
$$  \hspace{1cm} (2.26)

where $u_{ke}(x, t)$ is the solution of (2.9b), (2.10), and (2.24), and $u_{k}(x, t)$ represents the solution of (2.9b), (2.10), and (2.12) (as shown in [3], also in [5, Section 9.4; 16].

3. Finite element discretization and error estimation

In order to discretize the nonlinear problem (2.9b), (2.10), and (2.24): find $u_{ke}(x, t) \in L^\infty([0, T], H^{-1,2}(\Omega_k))$, such that
\[ \frac{\partial u_{k\varepsilon}}{\partial t} - \mathcal{A} u_{k\varepsilon} + g_{\varepsilon}(u_{k\varepsilon}) = 0 \quad \text{on } \Omega_k \times (0, T], \] (3.1a)

\[ u_{k\varepsilon}(x, 0) = \psi(x) \quad \text{on } \Omega_k, \] (3.1b)

\[ u_{k\varepsilon}(x, t) = \psi(x) \quad \text{on } \Gamma_k \times (0, T], \] (3.1c)

where

\[ g_{\varepsilon}(u_{k\varepsilon}) = \frac{1}{\varepsilon} \min_{\Omega_k}(u_{k\varepsilon} - \psi, 0), \] (3.2)

we use a method of lines formulation, as implemented in the finite element software system FEMLAB [4].

FEMLAB solves nonlinear time-dependent problems of the form

\[ d_a \frac{\partial u_{k\varepsilon}}{\partial t} + \Gamma_{i,j} = F \quad \text{on } \Omega_k \times (0, T], \] (3.3a)

\[ -n_i \Gamma_i = G + \frac{\partial R}{\partial u_{k\varepsilon}} \lambda_k \quad \text{on } \Gamma_k \times (0, T], \] (3.3b)

\[ 0 = R \quad \text{on } \Gamma_k \times (0, T], \] (3.3c)

with appropriate initial data for \( u_{k\varepsilon} \).

In (3.3), it is assumed that \( \Gamma, F \) can depend on \( u_{k\varepsilon}, \nabla u_{k\varepsilon}, x \) and \( t \), and that \( G, R \) can depend on \( u_{k\varepsilon}, x \) and \( t \) only.

The Dirichlet boundary conditions (3.1c) are replaced in FEMLAB by the generalized Neumann boundary conditions (3.3b) in order to simplify the variational formulation of (3.1). This is achieved with the introduction of the Lagrange multipliers \( \lambda_k \) in (3.3b). Corresponding to problem (3.1), the quantities that appear in (3.3) are

\[
\begin{align*}
& d_a = 1, \\
& \Gamma_i = -\frac{1}{2} \sigma_i^2 u_{k\varepsilon,x_i} - (r - \frac{1}{2} \sigma_i^2)u_{k\varepsilon}, \\
& F = -ru_{k\varepsilon} - \frac{1}{\varepsilon} \min(u_{k\varepsilon} - \psi, 0), \\
& G = 0, \\
& R = \psi - u_{k\varepsilon}.
\end{align*}
\] (3.4)

As usual, \( n_i \) represents the \( i \)th component of the outward unit normal vector to \( \Gamma_k \).

We obtain the variational form of (3.3) by multiplying (3.3a) by an arbitrary test function \( v \in \mathcal{H}^{-1,2}(\Omega_k) \), and (3.3c) by an arbitrary test function \( \mu \in \mathcal{H}^{-1,2}(\Gamma_k) \). Applying the divergence theorem and using (3.3b), we obtain

\[
\begin{align*}
& \left\{ \begin{array}{l}
\left( d_a \frac{\partial u_{k\varepsilon}}{\partial t} + \Gamma_{i,j}, v \right)_{\Omega_k} - (\Gamma_i, v_i)_{\Omega_k} - (F, v)_{\Omega_k} = \left( G + \frac{\partial R}{\partial u_{k\varepsilon}} \lambda_k, v \right)_{\Gamma_k}, \\
(R, \mu)_{\Gamma_k} = 0.
\end{array} \right.
\end{align*}
\] (3.5)
Also, multiplying (3.1b) by \( v \), we obtain
\[
(u_{k,e}(x,0), v)_{\Omega_h} = (\psi(x), v)_{\Omega_h},
\]
(3.6)

Let \( V_{h,k} \) be a finite dimensional subspace of \( \mathcal{H}^{-1/2}(\Omega_h) \) spanned by the functions \( \{ \phi_{h1}^k, \ldots, \phi_{hn}^k \} \). Specifically, we consider \( \phi_{hI}^k, I = 1, \ldots, N_p, \) as piecewise linear polynomials on a quasiuniform triangulation \( \tau_{hI} \) of \( \Omega_h \). Similarly, we take \( \mathcal{V}_{h,k} \) as a finite dimensional subspace of \( \mathcal{W}^{-1/2}(\Gamma_h) \), spanned by the set of functions \( \{ \psi_{h1}^k, \ldots, \psi_{hM_p}^k \} \). Again, \( \psi_{hI}^k, L = 1, \ldots, M_p, \) are piecewise linear polynomials on a quasiuniform triangulation \( \partial \tau_{hI} \) of \( \Gamma_h \). Let \( h_k \) denote the size (e.g. the length of the longest side) of \( \tau_{hI} \cup \partial \tau_{hI} \).

We approximate \( u_{k,e} \) and \( \lambda_k \) by
\[
U = \sum_{I=1}^{N_p} c_I(t) \phi_I(x),
\]
(3.7a)
\[
\lambda = \sum_{L=1}^{M_p} \lambda_L(t) \psi_L(x).
\]
(3.7b)

We intentionally dropped the indices \( k, e, h \) in the formulas above for notational clarity. We can infer, though, that \( c_I \) and \( \lambda_L \) depend on \( k, e, h \), while \( \phi_I \) and \( \psi_L \) depend only on \( k, h \).

By substituting (3.4) and (3.7) into the variational form (3.5) and (3.6), and also replacing \( v \) and \( \mu \) by \( \phi_I \) and \( \psi_L \), we obtain the finite element discretization of (3.1)
\[
DA\hat{c} + (CA + AL + A)c + F(c) + N^T \lambda = 0, \quad (\text{3.8a})
\]
\[
Nc = m, \quad (\text{3.8b})
\]
\[
DAc(0) = b, \quad (\text{3.8c})
\]

where \( c = \{ c_I \}_{I=1,\ldots,N_p}, \lambda = \{ \lambda_L \}_{L=1,\ldots,M_p} \) and
\[
\begin{align*}
DA_{\{I\}} &= (\phi_I, \phi_J)_{\tau_k}, \\
CA_{\{I\}} &= \frac{1}{2} \sigma_I^2 (\phi_I, \phi_J)_{\tau_k}, \\
AL_{\{I\}} &= (r - \frac{1}{2} \sigma_I^2) (\phi_I, \phi_J)_{\tau_k}, \\
A_{\{I\}} &= r (\phi_I, \phi_J)_{\tau_k}, \\
F(c)_J &= (g,c J)_{\tau_k}, \\
N_{\{L\}} &= (\phi_J, \psi_L)_{\partial \tau_k}, \\
m_L &= (\psi, \psi_L)_{\partial \tau_k}, \\
b_J &= (\psi, \phi_J)_{\tau_k}.
\end{align*}
\]
(3.9)

In order to eliminate the Lagrange multipliers as unknowns from (3.8) and to reduce the size of the system, we use the \textsc{Femlab}'s constraints elimination method. For this, let us introduce \( M \) as the matrix whose columns form a basis for the null space of \( N \), i.e.
Consider the decomposition
\[ c = Mc_m + c_d. \] (3.11)
After substituting (3.11) in (3.8b), we obtain
\[ Nc_d = m, \] (3.12)
which allows us to solve for \( c_d \). Neither \( N \) nor \( m \) do not depend on time, since the boundary conditions (3.1c) are independent of time. Therefore, \( c_d \) is independent of time. We use this property and substitute (3.11) in (3.8a). We obtain a system of nonlinear ordinary differential equations to determine \( c_m \)
\[ D_e \dot{c}_m + K e c_m + H(c_m) = -M^T K c_d, \] (3.13)
where
\[
\begin{align*}
K &= CA + AL + A, \\
D_e &= M^T DAM, \\
K_e &= M^T KM, \\
H(c_m) &= M^T F(Mc_m + c_d).
\end{align*}
\] (3.14)
From (3.9) and (3.14), we see that \( D_e \) is a symmetric and positive definite matrix; therefore, invertible, which allows us to write (3.13) in an explicit form
\[ \dot{c}_m = G(c_m, t), \] (3.15)
where \( G(\cdot, \cdot) \) can be readily obtained from (3.13). From (3.8c), we also find that
\[ D_e c_m(0) = M^T (b - DAC_d), \] (3.16)
which determines the initial conditions for the system (3.15).

The resulting system of ordinary differential equations (3.15) is stiff, and to solve it we chose MATLAB’s `ode15s` routine, which is a numerical differentiation formula (NDF) with quasiconstant step size and variable orders from 1 to 5.

For an equation of the form
\[ \dot{c} = G(c, t), \] (3.17)
the general (NDF) method of order \( k \) has the form
\[ \sum_{i=1}^{k} \frac{1}{i} \nabla^i c_{n+1} - iG(c_{n+1}; t_{n+1}) - \alpha_k h_k (c_{n+1} - c^0_{n+1}) = 0, \] (3.18)
where
\[ c^0_{n+1} = \sum_{i=0}^{k} \nabla^i c_n \] (3.19)
is the initial guess,
\[ \gamma_k = \sum_{j=1}^{k} \frac{1}{j}, \]  
(3.20)

\( l \) is the step size, and \( c_n \) is an approximation of \( c(t_n) \). Let \( c = \{c_i\}_{i=1,...,N} \).

If we use the identity
\[ c_{n+1} - c_{n+1}^0 = \nabla^{k+1} c_{n+1}, \]  
(3.21)

we see that the (NDF) formula (3.18) differs from a (BDF) formula only in the last coefficient \( \gamma_k \) \((-1)/(k+1) \) for a (BDF) discretization formula. The parameter \( \gamma_k \) is chosen to maximize the angle of \( A(x) \) stability.

It is shown [13] that the NDF method has a higher accuracy than the corresponding BDF of the same order (for \( k = 1,...,4 \)), with a slight (none for \( k = 1,2 \)) reduction of the region of \( A(x) \) stability. For \( k = 5 \), NDF and BDF methods coincide (\( \gamma_k = 0 \)).

By induction on \( k \), we may show that
\[ \sum_{i=1}^{k} \frac{1}{i} \nabla^i c_{n+1} = \gamma_k (c_{n+1} - c_{n+1}^0) + \sum_{i=1}^{k} \gamma_i \nabla^i c_n, \]  
(3.22)

Then (3.18) can be written as
\[ (1 - \gamma_k) (c_{n+1} - c_{n+1}^0) + \sum_{i=1}^{k} \gamma_i \nabla^i c_n - lG(c_{n+1}, t_{n+1}) = 0. \]  
(3.23)

This implicit nonlinear equation is solved by a Newton iteration, with the Jacobian provided analytically.

At each time step, the local error \( e_I = \{e_i\}_{i=1,...,N} \), can be controlled from the relation \(|e_i| \leq \text{reltol}|c_i| + \text{atol}_I\), by the user specified quantities atol and reltol.

It can be computed that, for the general (NDF) scheme of order \( k \leq 5 \) (3.18), the root condition is satisfied, i.e. all roots of the characteristic polynomial \( \rho(\xi) \) are either on the unit circle and simple or inside of the unit circle. Therefore, the scheme (3.18) is strongly stable [1, Section 5.2], and the accuracy of order \( k \) guarantees global convergence of order \( k \).

We are now able to formulate an error estimate for the fully discretized numerical solution of the penalized problem (3.1).

**Theorem 3.1.** Let \( u \in L^\infty([0,T], W^{-1,2}(\mathbb{R}^n)) \) be the exact solution of the variational problem (2.4b) and (2.11), and \( \{c_{n}^{k}\} \) be the space–time discretized finite element solution of the penalized problem (3.1). The following error estimate holds:

\[ \text{Error Estimate} \]
\[
\max_{t_n \in [0,T]} \| u(x_1, t_n) - c_{ln}^{k}\|_{L^\infty(\Omega)} \leq C(h_k^{-\mu} + l_k + \varepsilon) + o(\Pi^{-1}_k), \quad k \to \infty, \quad \varepsilon, h_k, l_k \to 0
\]

for \( \forall \mu > 0 \) and for \( \sqrt{G} \subset \Omega_k \), where \( \Pi_k \) is the radius of \( \Omega_k \) and \( l_k \) is the time step for each \( k \). Also, \( h_k^{-\mu} \) represents a correction to the finite element error due to the nonsmooth initial conditions (as shown in [5, Theorem 6.14]).

In the formula (3.24) above, the first term is due to the finite element piecewise linear approximation, the second term represents the time discretization error, as described before, and the third term is due to the penalization error (as per formula (2.26)). The last term is due to the artificial boundary conditions, as shown in (2.13).

**Proof.** Let \( u_k(x, t) \) be the solution of (2.9b), (2.10), and (2.12), \( u_{ke}(x, t) \) the solution of (3.1), and \( c_{I}^{ke}(t) \) the finite element approximation of \( u_{ke}(x_1, t) \), as per (3.7). By using (2.13) and (2.26), and the triangle inequality, we have

\[
\| u(x_1, t_n) - c_{ln}^{ke} \|_{L^\infty(\Omega)} \\
\leq \| u(x_1, t_n) - u_k(x_1, t_n) \|_{L^\infty(\Omega)} + \| u_k(x_1, t_n) - u_{ke}(x_1, t_n) \|_{L^\infty(\Omega)} \\
+ \| u_{ke}(x_1, t_n) - c_{I}^{ke}(t_n) \|_{L^\infty(\Omega)} + \| c_{I}^{ke}(t_n) - c_{ln}^{ke} \|_{L^\infty(\Omega)} \\
\leq o(\Pi^{-1}_k) + C(\varepsilon) + C(h_k^{-\mu}) + C(l_k) \quad \forall t_n \in [0, T].
\]

Applying maximum on \( t_n \) to (3.25), we obtain (3.24). \( \square \)

**Remark 3.1.** The numerical experiments in Section 4 concentrate on the effects of the finite element discretization error and on the error produced by the artificial boundary conditions. Therefore, we will supply the parameters \( \varepsilon \) and \( l_k \) or, alternatively, the local error tolerance in time

\[
tol = \max_{I=1,...,N} \left( atol_I, reltol |c_I| \right),
\]

such that their corresponding error contributions can be neglected. In our experiments we chose \( h_k \geq 0.008 \); hence we choose \( \varepsilon = 1e^{-5} \) and \( tol = 1e^{-8} \).

**Remark 3.2.** The error estimate (3.24) remains valid for the similar fully discretized finite element solution of the linear parabolic partial differential equation that characterizes the European case by dropping the \( \varepsilon \) term.

### 4. Numerical results

We verify the validity of the error estimate (3.24) for a variety of options on uncorrelated assets. Particularly, we will be interested in the error dependence
on three parameters of interest: the spatial mesh size \( h_k \), the radius of the computational domain \( P_k \) and the initial transient state parameter \( \epsilon \). The effect of the other two parameters from (3.24) is always eliminated, as per Remark 3.1. Specifically, we analyze European and American options on minimum and on the geometric average of two and three assets.

Since for all European option types discussed here have a known exact solution, we will use this solution in our asymptotic analysis for validating (3.24). No such solution is available for the American option problem (2.4b) and (2.11). In this case, we are able to reduce the \( n \)-dimensional variational problem (2.11) to a one-dimensional nonlinear partial differential equation, which we solve numerically with a sufficiently high accuracy, and consider this as an exact solution. This method is tested and validated for the simpler European option problem (2.4).

As a final remark, let us recall that an American call option with no dividends does not exercise early (see e.g. [15]). Therefore, this type of option is identical to an European call option. For this reason, no particular attention will be given in our study to the American call option.

In the following tests, we chose the computational domain \( \Omega_k \) (introduced in (2.9)), and the approximation domain \( \Omega_a \equiv \overline{G} \) (introduced in (2.13)), as

\[
\Omega_k = D_c \times \cdots \times D_c, \quad \text{where} \quad D_c = [\log(E) - R_c \cdots \log(E) + R_c], \quad \text{and} \\
\Omega_a = D_a \times \cdots \times D_a, \quad \text{with} \quad D_a = [\log(E) - R_a \cdots \log(E) + R_a].
\]

The following values of the other parameters were used (unless otherwise noted):

- the radius of the computational domain \( R_c \in \{0.4, 0.6, 0.8, 1, 1.5\} \);
- the radius of the approximation domain \( R_a = 0.1 \);
- the volatility for each asset \( \sigma_i = 0.3, \ \forall i = 1, \ldots, n \);
- the interest rate \( r = 0.05 \);
- the strike price \( E = 100 \).

For the asymptotic analysis that follows, we have chosen the expiration time \( T = 1 \) year.

The multivariate standard normal distribution function that appears in the exact solution formulas is computed using a FORTRAN implementation of the code presented in [12].
4.1. Asymptotic analysis

We analyze the rate of convergence of the numerical method described in Section 3 for a series of representative cases.

4.1.1. European call option on the minimum of \( n \) assets

As introduced in Section 2, the price of this option is characterized by the initial value problem (2.4), in which the payoff \( \psi \) is given by (2.6). As shown in (2.10), \( \psi(x) \) also represents the artificial boundary condition imposed on \( \Gamma_k \) for the numerical solution \( u \). The exact solution of (2.4) for \( n = 2 \) is given in this case by a generalization of the Black Scholes formula [14]. For \( n = 3 \), we use a similar analytical solution found in [8].

We solve (2.9) and (2.10) numerically on \( \Omega_k \) for different values of \( R_c \).

In Figs. 1a and 2a, we plot the relative maximum error on \( \Omega_a \) at \( T = 1 \), given by

\[
\text{ERR} = \frac{\|u(\cdot,1) - c^k\|_{L^\infty(\Omega_k)}}{\|u\|_{L^\infty(\Omega_k)}}
\]

versus the mesh size (\( h_k \)) and versus the degrees of freedom (dof) of \( \Omega_c \), for an increasing sequence of radii of the computational domain \( R_c \) and for \( n = 2 \). Due to the initial error at the boundary which propagates inside \( \Omega_a \) with time, the finite element rate of convergence is polluted. This pollution becomes less severe when \( R_c \) is bigger, as predicted by (2.13).

The corresponding convergence results for \( n = 3 \) appear in Figs. 1b and 2b and exhibit a similar behavior. As predicted by (3.24), the total error in the numerical solution decreases as \( k \to \infty \) and \( h_k \to 0 \).

4.1.2. European put option on the minimum of \( n \) assets

The European put option on the minimum of \( n \) assets is governed by the initial value problem (2.4), with a payoff given by (2.7). We consider \( \psi(x) \) to prescribe the artificial boundary conditions on \( \Gamma_k \) for \( u \), and solve (2.9) and (2.10) numerically on \( \Omega_k \). The exact solution of (2.4) and (2.7) for \( n = 2 \), is presented in [14].

In Fig. 3a and b, we show the relative maximum error on \( \Omega_a \), ERR, versus the mesh size (\( h_k \)) and versus the degrees of freedom of \( \Omega_c \) (dof), for an increasing sequence of \( R_c \)'s and for \( n = 2 \).

As for a call option, the artificial boundary conditions (2.10) pollute the finite element discretization error in the interior of \( \Omega_a \), at a rate proportional to \( R_c^{-1} \).

Again, the estimate (3.24) holds as \( h_k \to 0 \) and \( k \to \infty \).
4.1.3. European put option on the geometric average of \( n \) assets

The European put option on the geometric average of \( n \) assets is governed by the initial value problem (2.4) with a payoff given by (2.8). As with the previous models, \( \psi(x) \) represents the artificial boundary condition on \( \Gamma_k \) for the numerical solution \( u \). No exact solution has been found in the literature in this case.

In order to find the exact solution of (2.4) and (2.8), we introduce the new stochastic variable

Fig. 1. (a) European call option on the minimum of two assets. (b) European call option on the minimum of three assets.
\[ x = S_1 \cdots S_n, \quad (4.2) \]

in which \( S_i \) is the asset price defined by (2.1). Since each \( S_i \) satisfies the stochastic differential equation (2.1), we see that

\[ \frac{dx}{x} = \frac{dS_1}{S_1} + \cdots + \frac{dS_n}{S_n} = nr \, d\tau + \sum_{i=1}^{n} \sigma_i \, dB_i. \quad (4.3) \]
By applying the *no arbitrage* opportunities assumptions and *Itô Calculus* to a portfolio \( P_1 = V_1 - \Delta x \), we obtain the partial differential equation that characterizes the price of the option \( V_1(x, \tau) \)

\[
\frac{\partial V_1}{\partial \tau} + \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + nr x \frac{\partial V_1}{\partial x} - r V_1 = 0, \tag{4.4}
\]
where the index 1 is used to denote a one-asset case. To simplify (4.4) further, we make the change of variables

$$\begin{align*}
y &= \ln x, \\
t &= T - \tau,
\end{align*}$$

(4.5)

and use the notation $V_1(x, \tau) = u_1(y, t)$ to obtain

$$\frac{\partial u_1}{\partial t} - A_1 u_1 = 0,$$

(4.6)

where

$$A_1 u_1 = \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 \frac{\partial^2 u_1}{\partial y^2} + \left( nr - \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 \right) \frac{\partial u_1}{\partial y} - ru_1.$$  

(4.7)

From (4.2) and (4.5) we observe

$$y = \sum_{i=1}^{n} x_i,$$

(4.8)

where $x_i$ were introduced in Section 2.

Therefore, under our current assumptions, the partial differential equation (2.4a) is identical to the simpler one-dimensional partial differential equation (4.6) by using the change of variables (4.8).

The payoff condition (2.4b) is replaced for the one-dimensional problem by

$$u_1(y, 0) = \psi_1(y) = \max(E - e^{y/n}, 0).$$

(4.9)

The exact solution of (4.6) and (4.9) (and, implicitly, of (2.4) and (2.8)), is given in Appendix A.

In order to check the accuracy of the numerical method, especially since we will use the same method for the American options case, we solve (4.6) and (4.9) numerically and exactly (as presented in Appendix A), and compare these solutions with the numerical solution of (2.8)–(2.10). Since we want the numerical solution of (4.6) and (4.9) to mimic its exact solution, when compared to the numerical solution of (2.8)–(2.10), we accurately compute it for $R_c^l = 3$ and a mesh size $h = 0.001$.

In Figs. 4a and 5a, we show the relative error on $\Omega_a$, ERR, between the numerical solution of (2.8)–(2.10) and both the exact solution (labelled with versus exact), and the numerical solution (labelled with versus numerical) of (4.6) and (4.9), for $R_c \in \{0.6, 0.8, 1, 1.5\}$ and $n = 2$. We observe that the numerical solution of (4.6) and (4.9) represents an accurate approximation for the exact solution, since it produces the same values for the error, when compared with the numerical solution of (2.8)–(2.10). There is a similar behavior of the error as in the previous two cases, i.e. the error decreases with
decreasing the mesh size $h_k$ and increasing the radius of the computational domain $P_k$, as stated in (3.24).

The corresponding asymptotic graphs for $n = 3$ are presented in Figs. 4b and 5b. For these graphs, the comparison was made directly versus the numerical solution of (4.6) and (4.9) obtained for $n = 3$, $R_c^l = 7.5$ and $h = 0.001$. 

Fig. 4. (a) European put option on the geometric average of two assets. (b) European put option on the geometric average of three assets.
4.1.4. American put option on the geometric average of \( n \) assets

The American put option on the geometric average of \( n \) assets is characterized by the variational problem (2.4b) and (2.11), with a payoff given by (2.8). Eq. (2.11) is approximated by its simpler penalized form (2.24), taken for \( \varepsilon = 1e-5 \), as per Remark 3.1. No exact solution exists in this case for (2.4b) and (2.24). In order to find an accurate approximation for this exact solution, we reduce it first to its one-dimensional equivalent, after the model described in Section 4.1.3.
\[ \frac{\partial u_1^\varepsilon}{\partial t} - \mathcal{A}_1 u_1^\varepsilon + \frac{1}{\varepsilon} \min(u_1^\varepsilon - \psi_1, 0) = 0, \]  

such that

\[ u_1(y, 0) = \psi_1(y) \]

holds on \( \mathfrak{R} \). In (4.10) and (4.11), \( u_1, \mathcal{A}_1, \) and \( \psi_1 \) have the same meaning as in Section 4.1.3.

The exact solution of (2.4b) and (2.11) is then approximated by the numerical solution of (4.10) and (4.11), computed for \( \varepsilon = 10^{-5}, R_c^1 = 6 \) and \( h = 0.001 \) for \( n = 2 \), and for \( \varepsilon = 10^{-5}, R_c^1 = 7.5 \) and \( h = 0.001 \) for \( n = 3 \).

The truncated version of (2.4b) and (2.24) is given by (3.1). We solve (3.1) numerically for \( \varepsilon = 10^{-5}, R_c \in \{0.6, 0.8, 1, 1.5\} \), compare it with the numerical solution of (4.10) and (4.11) solved with the parameters given above, and plot the relative maximum error at \( T = 1 \) versus mesh size (\( h \)) in Fig. 6a for \( n = 2 \), and in Fig. 6b for \( n = 3 \). Again, the estimate (3.24) is valid.

While the solution of the penalized problem (2.24) converges to the solution of the variational inequality (2.12) as \( \varepsilon \to 0 \), the condition number of the stiffness matrix of the finite element discretization increases at a rate of \( O(1/\varepsilon) \). This would make the computation more difficult and less reliable as \( \varepsilon \) decreases.

### 4.2. The error in space

Let us consider the European call option on the minimum of two assets discussed in Section 4.1.1. We compute the pointwise relative error on \( \Omega_a \) at \( T = 1 \)

\[ \text{Perr}(x) = \frac{|u(x, 1) - e^k|}{\|u\|_{L^\infty(\Omega_a)}}, \]

and plot it on the approximation domain \( \Omega_a \).

Two representative graphs, for \( R_c = 0.8 \) and \( h_k = 0.09 \), and for \( R_c = 0.8 \) and \( h_k = 0.02 \), are presented in Fig. 7a and b. The two graphs capture the transition from the state of the error dominated by the finite element discretization term in (3.24) (Fig. 7a for \( h_k = 0.09 \)), and the state dominated by the artificial boundary term in (3.24) (Fig. 7b for \( h_k = 0.02 \)).

We remark in Fig. 7b that the error is maximum on the right boundary of the domain, since the absolute error

\[ \text{Berr}(x) = \max_{t \in [0,T]} |u(x, t) - \psi(x)|, \text{ defined for } x \in \Gamma_k, \]

which propagates inside \( \Omega_a \), is maximum on the right boundary of \( \Omega_k \).
4.3. Error evolution in time

In this section, we consider the European put option on the minimum of two assets, characterized by (2.4) and (2.7). We compute the relative maximum error between the the exact solution of (2.4), and the numerical solution of (2.9) and (2.10), ERR, computed for $R_c \in \{0.6, 0.8, 1, 1.5\}$ and a fixed spatial mesh.
size $h_k = 0.03$, at different time values $t \in [0, 1]$. We plot the relative error for $t \in [0 : 10^{-4} : 0.01]$ in Fig. 8, for $t \in [0 : 10^{-3} : 0.1]$ in Fig. 9a, and for $t \in [0 : 10^{-2} : 1]$ in Fig. 9b.
We distinguish three regimes:

- The first regime is given by the initial transient of the numerical solution which exhibits a rapid increase due to the nonsmooth initial data $u(x,0) = \psi(x)$ (see, for example, Fig. 8 for $t \in [0 \ldots 0.001]$, if $R_c = 0.6$).
- The second regime (e.g. $t \in [0.001 \ldots 0.5]$, if $R_c = 0.6$, as shown in Fig. 9a and b), is characterized by a diminishing of the initial transient. For the first two regimes, the error estimate
  \[ \|u - c_k\|_{L_1(\Omega_a)} \leq C h^{2-3\mu} t^{-2+\mu} \|\psi\|_{L^1(\Omega_a)} \quad \text{for } \forall \mu > 0, \forall t > 0 \]  
  (4.12)
  is valid (see [5, Theorem 6.14]). Except the initial decrease in error, the finite element discretization error is dominant in the second regime.
- The third regime is exhibiting another increase in the error, due to the artificial boundary condition $\psi(x)$ on $\Gamma_k$ (see $o(\Pi_k^{-1})$ term in (3.24)). As expected, the artificial boundary condition becomes dominant at an earlier time for a small $R_c$ ($t \approx 0.5$, if $R_c = 0.6$ and $t \approx 0.8$, if $R_c = 0.8$), and later for a bigger $R_c$.

4.4. Error dependence on $R_c$

In this section, we conduct a more detailed study of the error dependence on the radius of the computational domain $R_c$.

Using a maximum principle, Kangro and Nicolaides [9] obtained an a priori estimate for the absolute maximum error in $\Omega_a$.
AbsErr = \|u - u^k\|_{L^\infty(\Omega)};

caused by the artificial boundary condition (2.10) on \(\Gamma_k\) (see formula (22) in [9]).

The estimate is derived for the noncoercive form of the initial value problem (2.4), in which the state variable is represented by the stock price \(S_i\). We obtain the noncoercive form of (2.4) in our case with the change of variables

\[ S_i = e^{x_i}, \quad \forall i = 1, \ldots, n. \]  

(4.13)
In [9], exact boundary conditions were assumed on the left boundary and artificial boundary conditions were imposed on the right boundary. Therefore we will confine our attention to a call option, for which the absolute error attains its maximum value at the right boundary (see Fig. 7b). Although present, the left boundary error is always dominated by the error on the right boundary; thus, it can be neglected in our study.

The noncoercive form of (2.4) and (2.5) is given by

\[
\frac{\partial u}{\partial t} - \mathcal{A}^s u = 0 \quad \text{on} \quad (0, \infty)^n \times (0, T],
\]

\[
u(s, 0) = \psi(s) \quad \text{on} \quad (0, \infty)^n,\]

where \(\mathcal{A}^s\) represents the elliptic operator, degenerate for \(s = 0\)

\[
\mathcal{A}^s u = \sum_{i=1}^{n} \left[ \frac{1}{2} \sigma_i^2 \frac{\partial^2 u}{\partial s_i^2} + r s_i \frac{\partial u}{\partial s_i} \right] - ru.
\]

For the call option on the minimum of \(n\) assets that we consider here, the payoff introduced by (2.6) becomes

\[
\psi(s) = \max(\min(s_1, \ldots, s_n) - E, 0).
\]

We are interested in estimating the error between the solution of (4.14)–(4.16) and the solution of its corresponding truncated version

\[
\frac{\partial u_k}{\partial t} - \mathcal{A}^s u_k = 0 \quad \text{on} \quad \Omega_k^s \times (0, T],
\]

\[
u_k(s, 0) = \psi(s) \quad \text{on} \quad \Omega_k^s,
\]

such that the artificial boundary conditions

\[
u_k(s, t) = \psi(s) \quad \text{on} \quad I_k^s \times (0, T]
\]

are satisfied. Currently, the superscript \(s\) denotes that the state variable is given by the stock price \(S_s\).

In (4.17) and (4.18), the transformed computational domain \(\Omega_k^s\) is given by (4.1a) with

\[
D^s_c = [Ee^{-R_c} \cdots Ee^{R_c}]
\]

and, similarly, the transformed approximation domain \(\Gamma^s\) is given by (4.1c), with

\[
D^s_a = [Ee^{-R_s} \cdots Ee^{R_s}].
\]

Applying the theoretical estimate from [9] to the European call option on the minimum of two assets, we obtain
\[
\left\{ \begin{array}{l}
|u(s, t) - u_k(s, t)| \leq 2Ee^{R_c}e^{-\frac{|R_c - R_a|}{0.01}} \equiv \text{errbnd}(R_c, R_a, t), \\
\forall s \in \Omega_a \text{ and } \forall t \in (0, T),
\end{array} \right.
\]

where \( u \) represents the solution of (4.14)–(4.16), and \( u_k \) represents the solution of the truncated problem (4.17) and (4.18) for \( n = 2 \).

The problem of interest is to find, for an arbitrary given time \( t \) and an arbitrary given radius \( R_a \), a lower bound for the radius of the computational domain \( R_c \), which determines a certain value (let’s say 0.01), for the function \( \text{errbnd}(R_c, R_a, t) \), i.e. for which the artificial boundary condition (4.18) does not pollute with more than 0.01 the value of \( u_k \) inside \( \Omega_a \).

Theoretically, the value of \( R_c \) is obtained by solving, for the arbitrarily fixed values of \( R_a \) and \( t \), the quadratic equation \( \text{errbnd}(R_c, R_a, t) = 0.01 \).

Computationally, we calculate, for the given \( t \) and \( R_a \), the absolute error \( \text{AbsErr} \) on \( \Omega_a \), for an increasing sequence of \( R_c \)’s and a small enough mesh size, until \( \text{AbsErr} \) becomes smaller than the desired value of 0.01. The mesh size is considered to be small enough when the value of \( \text{AbsErr} \) does not change significantly with decreasing the mesh size (i.e. the behavior shown in Section 4.1).

The obtained values of \( R_c \) are then inverse interpolated by a spline, in order to obtain an approximation for the value of \( R_c \) which produces an absolute error on \( \Omega_a \) of 0.01.

In Fig. 10 we plot the theoretical and the computed values of \( R_c \), for \( R_a = 0.1 \) and 0.2, and for a sequence of times \( t \in [0.2 \ldots 0.8] \). We remark that the computed values of \( R_c \) respect the theoretical estimate provided by (4.21), and follow the same trend, thus validating this estimate. Although the mag-
nitude of $R_c$ obtained from the theoretical estimate (4.21) is approximately 1.5 times the magnitude of the computed value, (4.21) can certainly provide an useful estimate for $R_c$.

5. Conclusion

We have analyzed from a theoretical and a numerical perspective the method of lines finite element discretization for the partial differential equation that characterizes an European option, and for the penalized form of the variational problem that characterizes an American option.

A detailed analysis of the discretization error has been provided, including a space–time distribution of the error and the error dependence on the radius of the computational domain.

Appendix A

The exact solution of (4.6) and (4.9) can be obtained from the fundamental solution of the one dimensional heat equation. We have found that the exact solution of (4.6) and (4.9) is

\[
\begin{cases}
  u_1(y, t) = E e^{-\sigma \sqrt{t}} N(-d_1) - e^{\gamma - (1-n)\gamma^2 / 2\sigma^2} N(-d_2), \\
  d_1 = \gamma - n \ln E / \sigma \sqrt{t} + (k_1 - \frac{1}{2}) \sigma \sqrt{t}, \\
  d_2 = \gamma - n \ln E / \sigma \sqrt{t} + \left( \frac{1}{n} - \frac{1}{2} + k_1 \right) \sigma \sqrt{t}, \\
  k_1 = \frac{\alpha}{\sigma^2} \\
  \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.
\end{cases}
\]

(\text{A.1})

In the formula (A.1) above, $N(\cdot)$ represents the standard univariate normal distribution function, given by

\[
N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-s^2 / 2} ds
\]

Since (4.6) and (4.9) is equivalent with the $n$-dimensional problem (2.4) and (2.8) via (4.8), then (A.1) provides the exact solution of (2.4) and (2.8) also.

References